Journal of Engineering Mathematics, Vol. 8, No. 4, October 1974
Noordhoff International Publishing – Leyden
Printed in The Netherlands

Comments

ON "A NOTE ON THE FLOW IN A TRAILING VORTEX" BY K. K. TAM

I. H. HERRON, JR.

Department of Mathematics, Howard University, Washington, D.C. 20001, USA

(Received March 26, 1974)

SUMMARY
In a recent paper in this Journal (7, 1973, 1–6) Tam suggested that early work by Batchelor on the decay of a laminar trailing vortex was in error. Tam's claim was that the decay of axial velocity defect at the center of the vortex went as x^{-1} instead of $x^{-1}\log x$. The purpose of this note is to refute Tam's claim. A further term in the asymptotic expansion for the circulation in the vortex is also obtained.

A similarity solution for the steady flow far downstream in a laminar trailing vortex was given by Batchelor [1]. He began with the Navier–Stokes equations written in cylindrical polar coordinates (x, r, θ). The equations were linearized starting with two approximations which were basically of boundary layer type: (i) He took axial variations to be much smaller than radial variations such that

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial r}\,.$$

(ii) The axial component of velocity, u, was much greater than the radial component, v, so that

$$u \gg v\,.$$

A final approximation was that: (iii) A state had been reached such that the axial velocity component differed little from the free stream, U. Thus $|u-U| \ll U$.

A similarity parameter $\bar{\eta}$ was defined as

$$\bar{\eta} \equiv \frac{Ur^2}{4\nu x}\,, \tag{1}$$

where ν is the kinematic viscosity of the fluid, taken to be incompressible, with constant density ρ.

The equation of motion for the swirl velocity w was written in terms of the circulation C. This equation was integrated and the asymptotic solution as $x \to \infty$ given by

$$C = rw = C_0(1-e^{-\bar{\eta}})\,, \tag{2}$$

with C_0 being the circulation at large r, assumed to be non-zero.

The pressure field was found and written as

$$\frac{p_0-p}{\rho} = \frac{C_0^2 U}{8\nu x} P(\bar{\eta})\,, \tag{3}$$

where

$$P(\bar{\eta}) = \int_{\bar{\eta}}^{\infty} \frac{(1-e^{-\xi})^2}{\xi^2}\,d\xi\,.$$

With the preceding formulas, Batchelor constructed an asymptotic solution for the axial velocity. He found the solution by implicitly requiring another condition in addition to the more obvious boundary conditions of finiteness at $\bar{\eta}=0$ and vanishing velocity deficit as

$x \to \infty$ and $\bar{\eta} \to \infty$. This implicit condition is that the mass flux of the velocity deficit through any cross-sectional plane $x = \text{const.}$ be finite. Furthermore by considering the drag associated with the core of the trailing vortex, Batchelor was able to fully resolve the axial velocity component. He considered both vortices from a wing to obtain

$$u = U - \frac{1}{8\nu x} e^{-\bar{\eta}} \left\{ \frac{1}{\pi} \frac{D}{\rho} + C_0^2 \left(\alpha + \log \frac{xU}{\nu} - 2 \log \frac{sU}{\nu} \right) \right\} + \frac{C_0^2}{8\nu x} Q_2(\bar{\eta}) .$$

Here, D is the total drag on the wing, α is a positive constant close to 1, and s is the distance between the two trailing vortices far downstream. The function $Q_2(\bar{\eta})$ is given by

$$Q_2(\bar{\eta}) = e^{-\bar{\eta}} \int_0^{\bar{\eta}} \left(\frac{1 - e^t}{t} + P(t) e^t \right) dt .$$

Batchelor's asymptotic solution was criticized by Tam [2], who attempted to refute its validity on the grounds of non-uniqueness. Tam sought to reconcile the ambiguity he had created by writing the equations of motion in paraboloidal coordinates (ζ, η, θ) and solving them by a perturbation expansion. The solution which he obtained suffers from two defects.

Firstly, the method Tam used to attack the "arbitrariness" of Batchelor's solution may also be applied to his own. Secondly, the mass-flux deficit given by his solution is not finite.

We begin with the exact dimensionless equations of motion as given by Tam:

$$\zeta \frac{\partial^2 \psi}{\partial \zeta^2} + \eta \frac{\partial^2 \psi}{\partial \eta^2} = -\zeta\eta(\zeta+\eta) l , \tag{4a}$$

$$\zeta \frac{\partial^2 l}{\partial \zeta^2} + \eta \frac{\partial^2 l}{\partial \eta^2} + 2 \frac{\partial l}{\partial \zeta} + 2 \frac{\partial l}{\partial \eta} = \frac{-\partial(\psi, l)}{\partial(\zeta, \eta)} - T \frac{\Omega}{\zeta\eta} \left(\frac{1}{\eta} \frac{\partial \Omega}{\partial \zeta} - \frac{1}{\zeta} \frac{\partial \Omega}{\partial \eta} \right) , \tag{4b}$$

$$\zeta \frac{\partial^2 \Omega}{\partial \zeta^2} + \eta \frac{\partial^2 \Omega}{\partial \eta^2} = - \frac{\partial(\psi, \Omega)}{\partial(\zeta, \eta)} . \tag{4c}$$

The independent variables (ζ, η) are given by

$$\zeta = \frac{U}{2\nu} [(x^2 + r^2)^{\frac{1}{2}} + x] ,$$

$$\eta = \frac{U}{2\nu} [(x^2 + r^2)^{\frac{1}{2}} - x] .$$

Thus for $x \gg r$, to a first approximation, η and $\bar{\eta}$, defined in (1), are the same.

The line $\eta = 0$ corresponds exactly to $r = 0$. The dependent variables are the stream function

$$\psi = \frac{U}{2\nu^2} \psi^* ,$$

the swirl component of vorticity

$$l = \frac{2\nu^2 l^*}{U^3} ,$$

and the circulation

$$\Omega = \frac{\Omega^*}{C_0} .$$

The asterisks denote dimensional quantities.

The velocity components in this system are

$$u = \frac{1}{(\zeta(\zeta+\eta))^{\frac{1}{2}}} \frac{\partial \psi}{\partial \eta} , \quad v = - \frac{1}{(\eta(\zeta+\eta))^{\frac{1}{2}}} \frac{\partial \psi}{\partial \zeta} . \tag{5}$$

The only non-dimensional parameter appearing is a Taylor number, $T \equiv C_0^2 / 4\nu^2$.

As stated, the non-linear problem is elliptic. Thus boundary conditions involving ψ, $\partial\psi/\partial\eta$ and Ω must be given along $\zeta=0$, $\eta=0$ and at infinity to make the problem fully posed. This information is not completely available, so an asymptotic solution is sought for $\zeta \gg \eta$, $\zeta\to\infty$. Tam made a perturbation expansion about a free stream with a decaying vortex:

$$\psi(\zeta,\eta) \sim \zeta\eta+\psi^{(1)}(\zeta,\eta)+\dots, \tag{6a}$$

$$\Omega(\zeta,\eta) \sim \Omega_0(\eta)+\Omega^{(1)}(\zeta,\eta)+\dots. \tag{6b}$$

The boundary conditions he used were:

$$\Omega=\psi=0\,,\ \frac{\partial\psi}{\partial\eta} = \zeta-K_1 g_1(\zeta) \text{ at } \eta=0\,, \tag{7a}$$

$$\Omega\to 1\,,\ \frac{\partial\psi}{\partial\eta}\to\zeta \text{ as } \eta\to\infty\,. \tag{7b}$$

However, the above formulation is misleading. In equation (7a), $-K_1 g_1(\zeta)$ represents the velocity deficit on the axis. Tam stated it as a boundary condition but it is actually unknown. No conditions are given for $\zeta=\eta$ $(x=0)$ or $\zeta\to\infty$. This is so because under the assumption $\zeta\gg\eta$,

$$\frac{\partial}{\partial\zeta} \ll \frac{\partial}{\partial\eta}$$

and the equations are then parabolic.

The substitution of equations (6) into (4) yields the equations:

$$l^{(0)}=0\,, \tag{8}$$

$$\frac{\partial^2\psi^{(1)}}{\partial\eta^2} = -\zeta^2 l^{(1)}\,, \tag{9a}$$

$$\frac{\partial}{\partial\eta}\,\eta^2\left(\frac{\partial^3\psi^{(1)}}{\partial\eta^3}\right)+\frac{\partial}{\partial\eta}\,\eta^2\left(\frac{\partial^2\psi^{(1)}}{\partial\eta^2}\right)-\zeta\eta\,\frac{\partial^3\psi^{(1)}}{\partial\eta^2\,\partial\zeta} = -T\Omega_0'\Omega_0\,, \tag{9b}$$

$$\Omega_0''+\Omega_0'=0\,. \tag{10}$$

The solution to equation (10) is

$$\Omega_0 = 1-e^{-\eta}\,,$$

representing the asymptotic decay to a constant circulation.

Equations (9a, b) differ from Tam's equations [2; 22, 23] for at this point he had made a separation assumption $\psi^{(1)}(\zeta,\eta)=g_1(\zeta)\psi_1(\eta)$, requiring that $g_1(\zeta)=o(\zeta)$. With this restriction he reasoned $g_1(\zeta)$ must be a constant. He wanted to insure that the inhomogeneous term in equation (9b) be present. However, in doing so Tam used an approach for which he had criticized Batchelor in an earlier part of his paper. Tam had argued in section 2, of his work that $\psi^{(1)}(\zeta,\eta)$ satisfying equation (9b) might be written as:

$$\psi^{(1)}(\zeta,\eta) = \sum_{i=1}^{k} g_i(\zeta)\psi_i^{(1)}(\eta)+\psi_p^{(1)}(\zeta,\eta)\,. \tag{11}$$

Tam's idea was to show that any separated combination of k functions satisfying the homogeneous part of (9b) plus a particular integral would be a solution. He explicitly demonstrated two possible such solutions. He also produced the particular integral, a function of η only,

$$\psi_p^{(1)}(\eta) = \frac{T}{2}\int_0^\eta e^{-x}dx\int_0^x e^t P(t)dt\,. \tag{12}$$

$P(t)$ has the same previously defined functional form. The "boundary conditions" to be satisfied are:

$$\psi^{(1)}(0,\zeta)=0\,,\quad \frac{\partial\psi^{(1)}}{\partial\eta}(0,\zeta)=-K_1 g_1(\zeta)\,, \tag{13a}$$

$$\frac{\partial\psi^{(1)}}{\partial\eta}\to 0 \text{ as } \eta\to\infty\,. \tag{13b}$$

A similar ambiguity may be created in Tam's solution by writing the stream function as a different combination of separation functions, such as

$$\psi^{(1)}(\zeta,\eta)=\log\zeta\,\psi_{11}(\eta)+\psi_{12}(\eta)\,. \tag{14}$$

This particular choice may seem as arbitrary as Tam's choice. But it should be mentioned that Batchelor made this choice after integrating the equation for the axial velocity, the equivalent of equation (9b), over a cross-sectional plane.

Equations for ψ_{11} and ψ_{12} in (14) are found by substituting into equation (9b). When this is done and the boundary conditions are satisfied we obtain

$$\psi_{11}(\eta)=A(1-e^{-\eta})\,, \tag{15a}$$

$$\psi_{12}(\eta)=A\int_0^\eta e^{-\xi}\int_0^\xi \frac{e^t-1}{t}\,dt+B(1-e^{-\eta})\,, \tag{15b}$$

where A and B are as yet arbitrary constants.

From the results of Batchelor, writing (15) in cylindrical coordinates it is found that

$$A=\frac{-\Gamma}{8\nu^2}=-\tfrac{1}{2}T\,,$$

$$B=-\tfrac{1}{2}T\left(\alpha-2\log\frac{sU}{\nu}\right)-\frac{D}{8\pi\rho\nu^2}\,.$$

Then the velocity deficit at the center may be determined as

$$-K_1 g_1(\zeta)=A\log\zeta+B\,.$$

While from Tam's calculations the deficit is

$$-K_1 g_1(\zeta)=\text{constant}\,.$$

This calculation has shown that the solution of Tam is not "unique" in the sense in which he chose to define it.

It was mentioned earlier that Batchelor required the physically meaningful condition of a finite mass flux deficit for his solution. Consider the equation for the stream function [2, equation 27]. The particular integral for the flux deficit is the same as equation (12) above. Of interest is the limiting value

$$\psi_\infty^{(1)}\equiv\lim_{\eta\to\infty}\psi_p^{(1)}(\eta)\,.$$

Integrate (12) by parts to find

$$\psi_\infty^{(1)}=\frac{T}{2}\lim_{\eta\to\infty}\left\{(\eta-1)P(\eta)+P(0)e^{-\eta}-\int_0^\eta\left[\frac{(1-e^{-t})^2}{t^2}e^{-(\eta-t)}-\frac{(1-e^{-t})^2}{t}\right]dt\right\}. \tag{16}$$

Batchelor showed that for $\eta\to\infty$, $P(\eta)\sim\eta^{-1}$. The integrals may be performed over two-regions: $0\leqq t\leqq 1$ and $1\leqq t\leqq\eta$. (Since η is to be come large it may be taken greater than 1.)

Over the first interval the integrands are bounded and independent of η. Consider the first integral in (16) evaluated over the interval $1\leqq t\leqq\eta$. Expanded it may be written

$$-e^{-\eta}\int_1^\eta \frac{e^t}{t^2}-\frac{2}{t^2}+\frac{e^{-t}}{t^2}\,dt=-e^{-\eta}\left[\left(\frac{e^t}{t^2}+\frac{2}{t}-\frac{e^{-t}}{t^2}\right)\Big|_1^\eta+2\int_1^\eta\left(\frac{e^{-t}}{t^3}-\frac{e^t}{t^3}\right)dt\right]$$

$$\sim -\eta^{-2}+O(\eta^{-3}) \text{ as } \eta\to\infty\,.$$

The second integral in (16) when expanded becomes

$$\int_1^\eta \left(\frac{1}{t} - 2\frac{e^{-t}}{t} + \frac{e^{-2t}}{t}\right) dt$$

$$\sim \ln \eta - 2ei(1) + ei(2) \text{ as } \eta \to \infty .$$

Thus the mass flux as computed by Tam is logarithmically infinite.

Next, that author went on to compute a correction $\Omega^{(1)}$ to the circulation which he found to be independent of the mass flux correction $\psi^{(1)}$. This was so because the logarithmic term, given in equation (14), did not appear in his equation. The correct equation for $\Omega^{(1)}$ is

$$\eta\Omega^{(1)}_{\eta\eta} + \eta\Omega^{(1)}_\eta - \zeta\Omega^{(1)}_\zeta + \Omega^{(0)}_\eta \psi^{(1)}_\zeta = 0 . \tag{17}$$

To determine the asymptotic behaviour of $\Omega^{(1)}$ equation (17) is integrated from $\eta = 0$ to $\eta = \infty$. This is equivalent to Batchelor's procedure of integrating over a cross-sectional plane [1, p. 653]. One is then led to seek an analogous solution

$$\Omega^{(1)}(\zeta, \eta) = \frac{1}{\zeta}(\Omega_{11}(\eta) \log \zeta + \Omega_{12}(\eta)) ,$$

where the Ω_{1i} satisfy the boundary conditions

$$\Omega_{1i}(\zeta, 0) = 0 , \quad \Omega_{1i} \to 0 \text{ as } \eta \to \infty , \quad i = 1, 2 .$$

The solutions obtained are

$$\Omega_{11}(\eta) = \tfrac{1}{2}A\eta\, e^{-\eta} ,$$

$$\Omega_{12}(\eta) = C_2 \eta\, e^{-\eta} + \frac{A}{2}\eta\, e^{-\eta} \int_0^\eta \left(\frac{1 - t - e^{-t}}{t^2}\right) dt .$$

The constant A is the same as in equation (15). C_2, on the other hand, is indeterminate and must be given by the upstream conditions which do not enter explicitly here.

Acknowledgements

The author wishes to thank Professor P. G. Saffman who suggested this note and carefully read its contents.

This work was supported by the United States Air Force Office of Scientific Research AFOSR-71-2092, while the author was a Research Fellow at the California Institute of Technology.

REFERENCES

[1] G. K. Batchelor, Axial Flow in Trailing Line Vortices, *Journ. Fluid Mech.*, 20 (1964) 645–658.
[2] K. K. Tam, A Note on the Flow in a Trailing Vortex, *Journ. Eng. Math.*, 7 (1973) 1–6.